Rudiments of Logic and Algebra
Class 1

25 septembre 2012

Objective : Refresh high-school math skills ; Manipulate basic mathematic objects, namely numbers. Understand the sets of numbers $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. Using numbers properties to solve equations. Introduction to set theory, definition, notation, understanding the concept of finite and infinite sets.

1 Number sets

1.1 Natural numbers $\mathbb{N}$

"One, two, three, four".... That is the first mathematic your ever deal with. The counting numbers are the numbers the most "natural" because they are thought to be directly observable ("there are two dogs", "three books are there"...etc), hence we call the set $\{1, 2, 3....\}$ the set of natural numbers. For convenience, we will include 0 in this set because it has the same properties, hence we note the set of natural numbers $\mathbb{N} = \{0, 1, 2, 3....\}$.

The brackets "{" and "}" tell us to consider the collection of things between them as one object, namely **a set**. The symbol $\mathbb{N}$ is just an abbreviation to allow us to refer to this object concisely. The elements of a set are the "things" in the collection. The symbol "$\in$" is pronounced *'is an element of'*, so for example $21 \in \mathbb{N}$ is an abbreviation for the sentence *"Twenty-one is an element of the set of natural numbers."*. We will review this notation at the end of the class as we will introduce the branch of mathematics that deals with sets.

Why do we define a set for natural numbers ? Well, we define a set because these numbers share common properties. The same way we define mammals as animals sharing specific characteristics, natural numbers share properties, that is they behave the same way while we perform some operations on them. Given three natural numbers x, y and z[1], you are already familiar with the following operations :

	Addition	**Multiplication**
Neutral element	$x + 0 = x$	$1 \times x = x$
Commutative Law	$x + y = y + x$	$xy = yx$
Associative Law	$(x + y) + z = x + (y + z)$	$x(yz) = (xy)z$

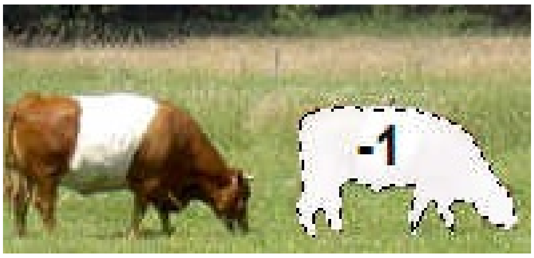
The **commutative law** say that we can interchange the order of the numbers we add or multiply ; for example $2 + 3 = 3 + 2$ and $2 \times 3 = 3 \times 2$. The **associative law** states that given

[1] It is a convention to use letters to speak about natural numbers, however the letter by itself doesn't mean anything and you will always have to precise $n \in \mathbb{N}$, if you want to say that n is a natural number.

three numbers, it does not matter whether we add or multiply the two first or the two last at the first place ; for example $(2+3)+1 = 2+(3+1)$ and $(2\times 3)\times 1 = 2\times(3\times 1)$. It seems that these laws are very basic but in fact, commutation and association do not always apply with elements other than numbers, you will see this in the linear algebra class later on when we will speak about matrices.

1.2 Integers $\mathbb{Z}$

Nowadays we use negative numbers as we use natural numbers, hence they seem perfectly natural for us. However, they may seem somehow less natural because it is difficult to observe -1 except if you are in a situation where you see one object missing from its usual place for example.



Besides this, extending the number system to include negative numbers seems to be quite essential in physics. For example we are standing on earth because there is the force of gravity F_{grav} that press us down and the reaction from the contact surface F_{norm} that prevent us from going in the center of the earth. This balance of forces is summarized by the equation $F_{grav} + F_{norm} = 0$ and because both of them are not null, this equation cannot be solved in $\mathbb{N}$.

Hence because having negative numbers make sense in many physical situations, we construct $\mathbb{Z}$ such that for every number a, there is a number b such that $a + b = 0$. We just extended our number system to a system where *subtraction* is always possible[2].

1.3 Rational numbers $\mathbb{Q}$

Rational numbers are commonly known as fractions. Why do we need fractions ? Consider that you have made 30 cookies, but that there are only 8 guests, how can they share equally the 30 cookies ? That's exactly the purpose of rational numbers and the meaning of the fraction $\frac{30}{8}$.

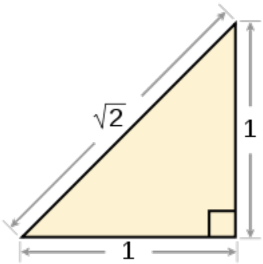
[2]you can notice that the commutative law is not applicable for subtraction

Hence to construct $\mathbb{Q}$, we add the following property to the existing number set $\mathbb{Z}$: *for every number $a \in \mathbb{Z}$ (except 0), there is a number $b \in \mathbb{Z}$ such that* $ab = 1$ (See [3]).

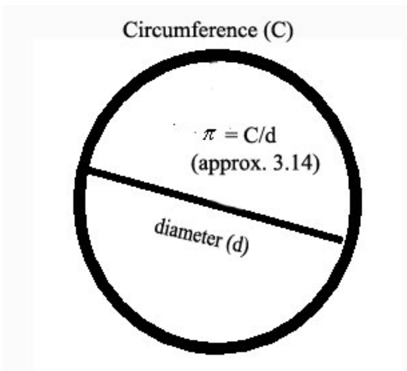
We call a and b **multiplicative inverses**. We just created a set where the *division* is always possible. As we will see in the exercises later, these numbers follow also the same rules as the integers (i.e addition, multiplication and subtraction). Note that to create $\mathbb{Z}$ and $\mathbb{Q}$ we just extended the existing rules of multiplication and addition that we had in $\mathbb{N}$ with two additional rules.

1.4 Real numbers $\mathbb{R}$

The set of real numbers include all numbers that can be represented by a sequence of decimals. Why do we need to extend again our number system ? Well consider a problem as follow : you want to define the side length of a square of area 2. As you know the area of a square is defined by multiplying the side of the square by itself. Hence if we note x the side, we have $x^2 = 2$. And it has been shown that actually you cannot represent x by a fraction ; it is said that x is *irrational*. Hence came the need of extending the number system such that we can include all the diagonals of any square in this system but also their sum, their multiplication and so on. That's what the set of real numbers is. We introduce a new symbol $\sqrt{}$ that we call *square root* and we say that in $x^2 = 2$, $x \pm \sqrt{2}$ and we say x is equal to plus or minus square root of 2.



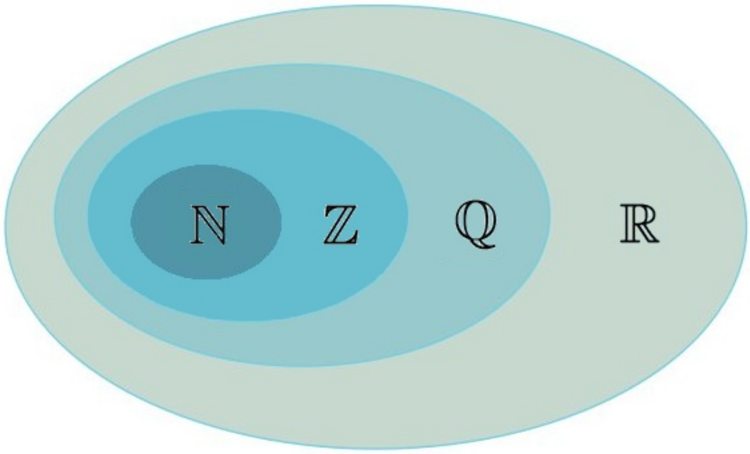
The idea with real numbers, is that you can represent any numbers with an *infinite* decimal representation. A well known real number is π : 3.141592653589793....



[3]You are strongly advised to read the chapter 2 of *A very short Introduction to Mathematics* by Tim Gowers (available on the course website) to understand why we don't have a multiplicative inverse for 0

An interesting thing to remember is that because real numbers require infinite storage, it is impossible to store them on a computer that can only store a fixed number of decimals. So keep in a mind that in fact while calculating expressions involving real numbers, the computer will give you only an approximation (though it is good enough). The calculation of the decimal of π became actually a contest among mathematicians that will get them busy until very long...

Now for the global picture, visually you can represent the set of numbers in the following hierarchy :



> **Exercise done in class**
>
> For each of the following numbers, list the sets in which it belongs :
> 1. π ? $\mathbb{R}$
> 2. $-\frac{3}{2}$? $\mathbb{Q}$, $\mathbb{R}$
> 3. $\sqrt{25}$? $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$
> 4. $\frac{\sqrt{52}}{\sqrt{13}}$? $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$

2 Operations on special numbers

So far we reviewed several types of numbers : the natural numbers, the integers, the rationals and the real numbers. However because we will soon manipulate these numbers to solve equations, it is important to be comfortable to add, subtract, multiply or divide them and this is what this section is about.

2.1 Fractions or Rational numbers

A fraction is essentially a division problem but not worked out. If we want to divide 3 by 2, we will preferably use the fraction $\frac{3}{2}$ than the number 1.5 because it is more convenient to

manipulate and to apply some operation on it before evaluating it.
When we add or subtract fractions, the usual way to do is to give a common denominator to both of them before adding the numerators. For example if I want to add $\frac{1}{8}$ and $\frac{2}{3}$:

$$\frac{1}{8}+\frac{2}{3}=\frac{1\times 3}{8\times 3}+\frac{2\times 8}{3\times 8}=\frac{3}{24}+\frac{16}{24}=\frac{19}{24}$$

The question is how do we found the common denominator ? For this, we choose a common **multiple** of both denominators. A common **multiple** is a number that can be divided by each two or more numbers without a remainder. For example 16 is a multiple of 8 as $16 = 8 \times 2$ but it is not a multiple of 3 because $16 = 3 \times 5 + 1$, i.e the remainder of the division of 16 by 3 is 1. At that point you might see that it is easy to find the common denominator of a fraction because if we multiply together the two denominators of each fraction such as in the previous example, we will obviously get a common multiple. That is, if I add two fractions $\frac{e}{a}+\frac{f}{b}$, $c = a \times b$ will always a common multiple of a and b.
However in some cases multiplying the denominators of each fraction together is not the simplest thing to do (although it is right !), consider the following addition :

$$\frac{1}{2}+\frac{2}{3}+\frac{5}{12}$$

It is true that $2 \times 3 \times 12$ is a common multiple of 2,3 and 12 and thus would be a suitable denominator. However, you might see that 12 is a common multiple of 3 and 2, hence it is sufficient to consider 12 as a common denominator.

$$\frac{1}{2}+\frac{2}{3}+\frac{5}{12}=\frac{1\times 6}{2\times 6}+\frac{2\times 4}{3\times 4}+\frac{5}{12}=\frac{19}{12}$$

When we multiply or divide fractions, you have been certainly told that you don't need to find a common denominator for this. As we said earlier a fraction is a division problem and a division is the inverse of the multiplication[4] (that is dividing by 2 is the same as multiplying by $\frac{1}{2}$). So when you multiply fractions it doesn't matter what you are doing first : dividing or multiplying because the order of operation is not important (that is $3\times 2\times 5$ is equivalent to $5\times 3\times 2$, i.e the commutative law). So the good new is that you simply need to multiply numerators together and denominators together :

$$\frac{4}{3}\times\frac{1}{5}=\frac{4\times 1}{3\times 5}=\frac{4}{15}$$

Dividing fractions is not so complicated as it may seem at first place. Dividing is the inverse of the multiplication, whenever we have a division, we can transform it into a multiplication. Recall that dividing by a number, say 2, is the same as multiplying by its inverse $\frac{1}{2}$. So when we divide by a fraction, say we divide a number (could be a fraction or not) by $\frac{2}{3}$ it is the same as multiplying this number by the inverse of our fraction that is $\frac{3}{2}$.

$$\frac{\frac{7}{5}}{\frac{2}{3}}=\frac{7}{5}\times\frac{3}{2}=\frac{21}{10}$$

[4]Remember that we are are in $\mathbb{Q}$ since we are dealing with fractions, we defined this set by extending $\mathbb{Z}$ with a single property : for every number $n \in \mathbb{Z}$ (except 0), there is a number $m \in \mathbb{Z}$ such that $nm = 1$ and thus $n = \frac{1}{m}$ and we call n and m multiplicative inverse.

Summary : Fractions

For a, b, c and $d \in \mathbb{R}$ with b, d and $c \neq 0$:

Addition/Soustraction :

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm cb}{bd}$$

Product :

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

Division :

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \times \frac{d}{c}$$

2.2 Exponents

What is an exponent ? An exponent is simply the symbol we use to say that we multiply a number by identical factors. $2 \times 2 \times 2 \times 2 \times 2$ is 5 times the same factor 2 hence we can simply write 2^5.
Now we can wonder about the properties of the exponent. An exponent is a number that belongs to the set of reals, hence it can be negative. What does it means to have a negative exponent ? It means that we multiply the inverse of a number by identical factors :

$$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = (\frac{1}{2})^3 = \frac{1}{2^3} = 2^{-3}$$

How do we manipulate numbers with exponents ? For example how do we multiply two numbers with exponent such as $9^5 \times 9^2$. You can write it as :

$$\begin{aligned} 9^5 \times 9^2 &= (9 \times 9 \times 9 \times 9 \times 9) \times (9 \times 9) \\ &= 9^7 \end{aligned}$$

Since the beginning we always defined exponents in $\mathbb{R}$, such that an exponent can also be a fraction or an irrational number. Well, we will not go in details for these cases but maybe it is interesting to know one more thing about exponent values, that is what happen when an exponent is 0 ? You may know already that for all $x \in \mathbb{R}$ with $x \neq 0$, $x^0 = 1$. And that might be surprising at first. Recall that we defined we have the following property in our number system : for every number a, there is a number b such that $a + b = 0$[5]. Hence, we can choose a and $b = -a$ in $\mathbb{R}$ to satisfy this equation. Let us rewrite our power 0, for all $x \in \mathbb{R}$ with $x \neq 0$:

$$\begin{aligned} x^0 &= x^{a-a} \\ &= x^a \times x^{-a} \\ &= \frac{x^a}{x^a} \text{ by definition of } x^{-a} \\ &= 1 \text{ for all } x \text{ because we defined } x \neq 0 \end{aligned}$$

What is important to understand here is that the properties we defined to extend our number system have important consequences when we manipulate numbers in general.

[5] It is the property we defined to justify the set of integers $\mathbb{Z}$

Summary : Exponents

For a, b and $x \in \mathbb{R}$ and $x \neq 0$:

Negative Exponent :

$$x^{-b} = \frac{1}{x^b}$$

Product :

$$x^a \times x^b = x^{a+b}$$

Division :

$$\frac{x^a}{x^b} = x^{a-b}$$

2.3 Squares Root

Every **positive** real number a has two square roots : $\sqrt{a}$ and $-\sqrt{a}$. For example the square roots of 9 are -3 and 3 because $-3 \times -3 = 9$ and $3 \times 3 = 9$. Square root of 9 or $\sqrt{9}$ is what we call a perfect square because it is equal to an integer ($\mathbb{Z}$). On the contrary $\sqrt{2}$ cannot be reduced to an integer, we saw already that $\sqrt{2}$ is irrational (that is why we created square root after all!).

Roots are the opposite operation of applying powers to a number. For example : $2^2 = 4$ and $\sqrt{4} = 2$. As you can raise power to cube, fourth power and so on, it is similarly possible to take the cube root of a number, the fourth root...etc. When we apply a root other than the square root, we indicate it by adding a number to the square root symbol. For example, if we want to calculate the cube root of 8, we wrote : $\sqrt[3]{8}$ and we know that $\sqrt[3]{8} = 2$ because $2^3 = 8$.

A few things to know while manipulating square roots :

Summary : Square roots

For a, $b \in \mathbb{R}$ with $a > 0$ and $b > 0$:

Identity :

$$\sqrt{a} \times \sqrt{a} = a$$

Addition :

$$b\sqrt{a} + c\sqrt{a} = (b+c)\sqrt{a}$$

Addition is possible only while adding square roots of the same number.

Product :

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

Division :

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

Using these properties allow you to simply square roots expressions by just separating anything that is a "perfect square" ; for example :

$$\sqrt{75} = \sqrt{25}\sqrt{3} = 5\sqrt{3}$$

Exercise done in class

Simplify A :

$$A = \sqrt{490}$$
$$A = \sqrt{49 \times 10}$$
$$A = 7\sqrt{10}$$

3 Equations

Now that we reviewed the different kind of numbers, we are ready to use them to solve equations. You spent certainly a lot of time in classroom in high-school to solve equations and you were certainly taught that the most important thing when we have an equation is to find a solution. However many equations of real life problems cannot be solved and if they can, then you might use a powerful computer to solve them at our place. Given this, why do we again speak about solving equations ? What is really important about equations is that they provide a precise way to describe various elements of the world and that working on solving them help us to actually understand what they mean. But first we need to be comfortable with manipulating numbers and that what this class is about as you will have ample time to review and practice the basics about equation solving.

3.1 Linear Equations

By definition, an equation is a mathematical statement that asserts the equality of two expressions. Linear equations are the simplest type of equation you will find as they involve only a single variable x in the first power, that is they don't involve any x^2 or x^3 terms. Linear equations are of the form : $ax + b = 0$ where $a \neq 0$. Contrary to $ax + b = 0$, note that $ax + b$ is not an equation but an algebraic equation because there is no equality in it. Solving an equation means to find all the solutions x that verifies the equality.

3.1.1 Solving equation by collecting terms each side of the equation

Example : Solve the following equation for $x \in \mathbb{R}$

$$2x - 3 = x + \frac{7}{2}$$

$$2x - x = \frac{7}{2} + 3$$
$$x = \frac{13}{2}$$

To solve linear equations, the idea is to isolate x on one side of the equality and to form an expression with the non-variable terms on the other side. How we do that ? We add, subtract, multiply and divide both sides of the equation by numbers and variables. It is important to keep in mind what an equation is, i.e a statement of equality, that is whatever manipulation you do on one expression or one side of the equation, you have to do **exactly** the same on the other.

Notice that in the previous example we did not have exactly an equation of the form $ax+b=0$, however the equation could have been written in this form since it is a linear equation.

$$2x - 3 = x + \frac{7}{2}$$

$$2x - x - 3 - \frac{7}{2} = 0$$

$$x - \frac{13}{2} = 0$$

So here is our form $ax+b=0$. You don't have to transform the equation this way each time you are solving. However notice that with this form it is easy to find the solution of the equation $\frac{13}{2}$. We have actually a nice systematic solution for x in $ax+b=0$:

Summary : Linear Equations

In the case of linear equation of the form $ax+b=0$ where $a \neq 0$ there is one unique solution $x \in \mathbb{R}$, namely $x = -\frac{b}{a}$.

One more step-by-step example : Solve in $\mathbb{R}$

$$2x + 3 = 6 - (2x - 3)$$

First remove the bracket to the right by paying attention to the sign :

$$2x + 3 = 6 - 2x + 3$$

Group variable terms at one side of the equation and non-variable terms at the other side :

$$4x = 6$$

So that :

$$x = \frac{6}{4}$$

And simplify :

$$x = \frac{3}{2}$$

3.1.2 Solving equation by expanding terms and collecting them

Let us start with a step-by-step example :

$$(x+1)(2x+1) = (x+3)(2x+3) - 14$$

We start by expanding the factors (hence, removing the brackets) :

$$2x^2 + x + 2x + 1 = 2x^2 + 3x + 6x + 9 - 14$$

Notice that we have terms in x^2 but it doesn't matter here since we can subtract them from both sides of the equation :

$$x + 2x + 1 = 3x + 6x + 9 - 14$$

Now proceed as previously and group terms each side of the equation :

$$x + 2x - 3x - 6x = 9 - 14 - 1$$

$$x = 1$$

Check that $x = 1$ is the solution of the equation by substituting it back in the original equation. For the left side we have :

$$(x + 1)(2x + 1) = (1 + 1)(2 + 1) = (2)(3) = 6$$

And for the right side :

$$(x + 3)(2x + 3) - 14 = (1 + 3)(2 + 3) - 14 = (4)(5) - 14 = 20 - 14 = 6$$

So both sides are equal to 6, $x = 1$ is the solution of the equation.

The idea is to get rid of any parenthesis and for this, we just have to use one property : **the distributive law** :

Distributive property

$$a(b + c) = ab + ac$$

And that is basically all you need to do a proper expansion and to simplify an equation.

More Examples done in class : Solve the following equation for $x \in \mathbb{R}$:

Exercise done in class

$$2(3x - 1) = -4x - (x + 1)$$
$$6x - 2 = -4x - x - 1)$$
$$11x = 1)$$
$$x = \frac{1}{11}$$

Exercise done in class

$$\frac{3}{5}x - 2 = \frac{1}{2}(3x - 5)$$
$$\frac{3}{5}x - 2 = \frac{3}{2}x - \frac{5}{2})$$
$$\frac{9}{10}x = -\frac{1}{2})$$
$$x = -\frac{5}{9}$$

Exercice

$$\frac{\sqrt{5}x}{\sqrt{5} - \sqrt{3}} - \frac{\sqrt{3}x}{\sqrt{5} + \sqrt{3}} = 1$$
$$4x = 1$$

3.2 Solving Equations involving Fractions

We are now dealing with equations where the variable is in the denominator such as :

$$\frac{1}{x} = \frac{2}{1+x}$$

At first sight appear not to be linear equations. However, with some algebraic manipulation they can be recast in a more familiar form.
The first thing to do here before rushing to the solution is to precise the constraints on the variable x. Because a denominator cannot be null, it follows that $x \neq 0$ and $1 + x \neq 0$. Having defined this, we clear the denominator from the variable to solve the equation :

$$1 + x = 2x$$

$$x = 1$$

Check that the solution obtained does follow the constraints on x previously defined. In this case, the solution of the equation is $x = 1$.

Exercise done in class

Solve the following equation

$$\frac{2}{x-1} + 3 = \frac{4x}{x-1}$$

with $x \neq 1$

$2 + 3(x - 1) = 4x$

$x = -1$

Summary

1. Define the constraints on x given the expression of the denominator(s)
2. Clear the denominator from the variable
3. Solve the equation
4. Check that the solution follows the constraint defined in 1)

3.3 Quadratic Equations**

A quadratic equation or second degree polynomial expression is an equation containing a squared variable x^2. Generally the quadratic equation is of the form :

$$ax^2 + bx + c$$

Where a, b and c are real numbers and $a \neq 0$ (otherwise this is simply a linear equation).

Do we really model real world problems with these equations ? The answer is obviously yes. Think about the trajectory of a ball that you just threw, that is a parabola. Well, quadratic equations are indeed describing parabolas. More on this during the lecture about functions. Just remember that behind equations there is always a real-world problems. Eventually equations get more and more complicated but that's the way real life is.

How do we solve quadratic equations? We do have an analytic formula for that, but sometimes it is also less costly to express the quadratic expression into linear factors (hence $ax+b$) with which we are already comfortable with. Both methods are explained below.

3.3.1 Solving equations using zero-product property

An equation such as $(x-1)(x+2)=0$ is a product of linear factors. The good news is that although this is a quadratic equation [6] the solution is easily obtainable with the help of a simple property.
Recall that a product of factor is equal to zero if at least one of its factor is equal to zero. In the example $(x-1)(x+2)=0$, it means that $x-1=0$ or $x+2=0$. Whenever you encounter such equation, you can use this zero-product property, namely :

Zero-product property

If $ab=0$ with a and b real numbers, then $a=0$ or $b=0$.

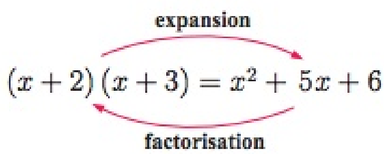
Exercise done in class

Solve the following equation :

$$(x+1)x(x-1)=0$$
$$x+1=0 \text{ or } x=0 \text{ or } x-1=0$$
$$x=-1 \text{ or } x=0 \text{ or } x=1$$

3.3.2 Factorization : Find a common factor

Because it is quite rare to have a ready made expression expressed in a product of factors such as $(x-1)(x+2)$, you will have sometimes to arrange the expression as so in order to solve it easily using the zero-product property.
Factorization is the process of writing an expression as a product of factors, it is the reverse process of expanding an expression.



One way of doing is to find a common factor between the term composing the sum. For example in x^2+3x is the sum of x^2 and $3x$ and x is said to be a common factor of both x^2 and $3x$:

$$x^2+3x = x \times x + 3 \times x$$
$$= x(x+3)$$

[6]You can try to develop it to check it! (*Solution :* x^2+x-2

Factorizing is useful to quickly detect solutions for an equation. For example, finding the solution of $x^2 + 3x = 0$ is easy given the fact that $x^2 + 3x = x(x + 3)$: write $x(x + 3) = 0$ such that we have $x = 0$ or $x = -3$ following the zero-product property reviewed before.

Exercise done in class

Factorize then solve the following equation :

$$(x-5)^2 = 2(x-5)$$
$$(x-5)(x-5) - 2(x-5) = 0$$
$$(x-5)[(x-5) - 2] = 0$$
$$(x-5)(x-7) = 0$$
$$x - 5 = 0 \text{ or } x - 7 = 0$$
$$x = 5 \text{ or } x = 7$$

So factorization is a powerful tool to easily solve an equation. However the main difficulty remain in finding a common factor. Sometimes it is obvious as in the previous example, but sometimes it needs some extra rearrangement of the expression. For example :

$$(x+2)^2 + 2x + 4$$
$$(x+2)(x+2) + 2(x+2)$$
$$(x+2)[(x+2) + 2]$$
$$(x+2)(x+4)$$

Exercise done in class

Factorize the following expression :

$$(x-3)^2 - x + 3$$
$$(x-3)(x-3) - (x-3)$$
$$(x-3)[(x-3) - 1]$$
$$(x-3)(x-4)$$

3.3.3 Factorization : using common square expansions *

First let us refresh our ideas on some common factorization and expansions of expressions. Reckon that :

Common factorizations/Expansions

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a-b)(a+b) = a^2 - b^2$$

Now let us consider a quadratic expression such as $A = 4x^2+4x+1$, in this case it is possible to find a and b such that $a = 2x$ and $b = 1$. Hence $A = (2x+1)^2$. So if we want to to solve $A = 0$, then the factorization gives us the solution very quickly $x = -\frac{1}{2}$.

Factorize A :

Exercise done in class

$A = x^2 + 6x + 9$
$A = x^2 + 2 \times x \times 3 + 3^2$
$A = (x+3)^2$

Exercise done in class

$A = x^2 - 7$
$A = (x - \sqrt{7})(x + \sqrt{7})$

However even if this little trick is sometimes useful to solve quadratic expression without going for the analytical solution, it is not necessary that the quadratic expression can be factorized into a perfect square. In this cases you should go for the analytic solution described in the following.

3.3.4 Analytic solutions *

Solving quadratic equation means that we want to know the values of x when $ax^2+bx+c = 0$. For this we have a analytic formula that work for every a,b and c : [7]

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For these solutions to exist in $\mathbb{R}$, then $b^2 - 4ac$ must be **positive** as it is under a square root because in $\mathbb{R}$, there is no square root of negative numbers. Then before hurrying to compute a solution, check the value of $b^2 - 4ac$;

- if $b^2 - 4ac < 0$ then there is no solution of the equation in $\mathbb{R}$.
- if $b^2 - 4ac \geq 0$, then there is two solutions in $\mathbb{R}$: $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

Note that when $b^2 - 4ac = 0$ there is just a single solution $x = \frac{-b}{2a}$. We call the solutions x the **roots** of the polynom.

Once you calculated the roots of the polynom, you can easily factorize the quadratic expression. When you have two distinct roots x_1 and x_2 for $ax^2 + bx + c = 0$ then :

$$ax^2 + bx + c = (x - x_1)(x - x_2)$$

[7] If you want to persuade yourself that this formula doesn't come from nowhere, you can do yourself the proof. Take $ax^2 + bx + c = 0$, multiply each side of the equation by $4a$ and add b then use what you know about factorization to come up with the formula. If any doubt come to ask us.

Note that when you have a single root x_1 then :

$$ax^2 + bx + c = (x - x_1)^2$$

The expression is a perfect square [8].

4 Introduction to set theory

Objective : In this part, we will learn elementary set theory and the basic properties of sets. We will also learn how to define the size of a set, and how to compare different sizes of sets. This will lead us to the notions of finite and infinite sets.

4.1 Notations

Without defining them rigorously, we saw already four simple sets :
- The natural numbers $\mathbb{N} = \{0, 1, 2, 3....\}$
- The integers $\mathbb{Z} = \{-3, -2, -1, 0, 1, 2, 3....\}$
- The rational numbers $\mathbb{Q} = \{\frac{a}{b} | a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0\}$
- The real numbers $\mathbb{R}$

But examples of sets can be found everywhere around us. For example, you can think about the set of all living beings, the set of European countries, the set of letters in your alphabet...etc. Each living being is an **element** of the set of all living beings, similarly each letter is an **element** of the set of letters.
If S is a set and s is an element of S, then we write $s \in S$. If it so happens that s is not an element of S, then we write $s \notin S$. If S is the set whose elements are A, B, and C, then we write $S = \{A, B, C\}$. The left brace and right brace visually indicate the bounds of the set, while what is written within the bounds indicates the elements of the set.
For example : if $S = \{1, 5, 9\}$ then $1 \in S$, but $7 \notin S$.

Sets are defined by their elements. The order in which the elements of a given set are listed does not matter. For example, $\{1, 2, 3\}$ and $\{3, 1, 2\}$ are the same set. It also does not matter whether some elements of a given set are listed more than once. For instance : $\{1, 1, 1, 2, 1, 1, 1, 1, 2, 2, 3\}$ is still the set $\{1, 2, 3\}$.
The elements of a set can be other sets. For example, $\{1, \{5\}\}$ is the set whose elements are 1 and $\{5\}$. So $1 \in \{1, \{5\}\}$ and $\{5\} \in \{1, \{5\}\}$, but $5 \notin \{1, \{5\}\}$.
The empty set, $\emptyset$, is a special set which doesn't have any elements ; in other words, $= \{\emptyset\}$. One can think of the empty set as a box with nothing inside.

A set might be defined by a property, for example it is possible to define the set of all vowels letters, the set of all integers greater than 4 and so on. If we note a property as $P(s)$ then $A = \{s \in S | P(s)\}$ means that the set A consists of elements s of S having the property $P(s)$. The colon : | is commonly read as "such that". Sometimes you can also see the notation with a semicolon : $\{s \in S : P(s)\}$ depending on the author.
Let us clarify this notation through an example. We saw that $\mathbb{N}$ was the set of natural numbers. If I want to put in a set A all natural numbers whose square value is equal to 1. If I note n a number, and the property $P(n)$ as $n^2 = 1$, I can write $A = \{n \in \mathbb{N} | n^2 = 1\}$.

[8] Note that here we are in the case where we could have used the technique in Section 3.3.3 to solve the equation

So until now we saw two things, we saw that we can construct sets of anything (even set of sets!) and that sets are defined by the elements their are composed with and sometimes by a property.

4.2 Relations between sets

Given some sets, we are interested now to define how are they related. Let us start with an example and consider the three following sets :

$$M = \{\text{all the mammals}\}$$
$$D = \{\text{all the dogs}\}$$
$$d = \{\text{my dog Bonnie}\}$$

We can try to figure out the relationships between M, D and d. For this we can use Venn diagrams, that is we draw M as a big circle with D as a smaller circle inside it and d as a single point inside D. Hence d is in B and d is in M, this reflects that my dog Bonnie is both a dog and a mammal and we know already how to reflect this relation with the symbol $\in$ ($d \in M$ and $d \in D$). We have also D inside M but here the notion of 'inside' is different, D is not in M in the sense of being a point in a circle (in the sense of d being in D), because the set D is not an element of M as the set of all dogs is not a mammal. However we can say that **every element in D is an element in M**. In that case we say that D is a **subset** of M and we note :

$$D \subseteq M$$

Here we can even note $D \subset M$, because it is easy to find an element of M that is not in D (for ex, a cat). If we have an other set R of all reptiles then we can write $R \nsubseteq M$ because R is not a subset of M. However if we define the set S of all sets of mammals, because this time a set of mammals is an element of S, we can write $D \in S$.
Although the membership relation $\in$ and the subset relation $\subset$ are related to each other, they behave quite differently as you will explore in the exercises.

A few more things about the subset relation $\subset$. Note that :
- The $\emptyset$ is included in every set A : $\emptyset \subseteq A$
- Every set A contains itself : $A \subseteq A$

Exercise done in class

True or false ?

1. $\emptyset = \{0\}$
2. $x \in \{x\}$
3. $\emptyset = \{\emptyset\}$
4. $\emptyset \in \{\emptyset\}$

Notice that $\emptyset$ and $\{\emptyset\}$ are quite different sets. $\emptyset$ is the empty set : it has **no** members. $\{\emptyset\}$ is a set which has **one** member. Hence what is the case here is that

$$\emptyset \neq \{\emptyset\}$$
$$\emptyset \in \{\emptyset\}$$

4.3 Set Equality

Intuitively two sets A and B are equal if they consist of the same elements and we write $A = B$. When two sets A and B are not equal, we note $A \neq B$.
So for example, if $A = \{1, 2, 3\}$ and $B = \{1, 2, 3\}$, we have $A = B$ that seems quite obvious.

Exercise done in class

However what do you think about :

$$A = \{\text{the purple-haired rabbits}\}$$
$$B = \{\text{the green-haired rabbits}\}$$

Do we have $A = B$?

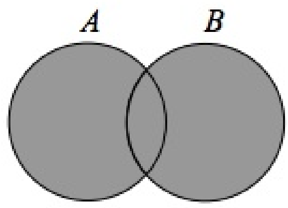
4.4 Set Operations

We just saw in this class that we can add, subtract, multiply and divide numbers to create new numbers. In the same way, several operations have been defined to create new sets from other sets by taking their *union*, their *intersection* or their *complement*.

Given two subsets A and B of E :

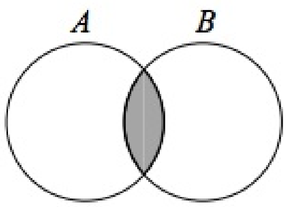
The *union* of two sets A and B is the set whose elements are all of the elements of A and B. We denote this operation by $\cup$. Formally we write :

$$A \cup B = \{x \in E | x \in A \text{ or } x \in B\}$$



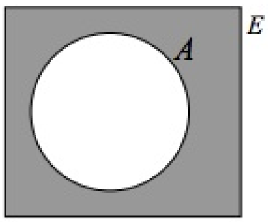
The *intersection* of two sets A and B is the set whose elements are both in A and in B. We denote this operation by $\cap$. Formally we write :

$$A \cap B = \{x \in E | x \in A \text{ and } x \in B\}$$



The *complement* of a set A is the set whose elements are not in A with respect to a set E. We note the complement of A : $\complement_E(A)$ or A^c . Formally we write :

$$\complement_E(A) = \{x \in E | x \notin A\}$$



> **Exercise done in class**
>
> Let $U = \{1, 2, 3, 4, 5, 6\}$, $X = 1, 5$, $Y = \{1, 2, 4\}$, $Z = \{1, 2, 4\}$
> Calculate the following sets :
> $X \cap Y^c = \{5\}$
> $(X \cap Y) \cup Y^c = \{1, 3, 5, 6\}$
> $X \cup (Y \cap Z) = \{1, 2, 5\}$
> $(X \cup Y) \cap (X \cup Z) = \{1, 2, 5\}$
> $(X \cup Y)^c = \{3, 6\}$

4.5 Cardinality *

Here we are dealing with cardinality of a set, that is to say the number of elements in a set. At first it seems a really simple concept. For example, given a set $A = \{1, 2, 3\}$, there are 3 elements in A therefore we can say that the cardinality of A is 3 and we note $Card(A) = 3$. If we have a set B such that $B = \{a, b, c\}$ then we can say that $Card(B) = Card(A)$. Note however that $A \neq B$.
Hence, cardinality is an easy concept to grasp for small and finite sets. But we reviewed some very interesting sets at the beginning of this class which we told to be *infinite* : the set of natural numbers $\mathbb{N}$, the set of the integers $\mathbb{Z}$, the set of rational numbers $\mathbb{Q}$ and the set real numbers $\mathbb{R}$. Intuitively we might feel that even if we are dealing with infinite sets, their degree of inifinity

must somehow be different.

Let us first consider the set of natural numbers. In a certain sense you can count the elements of $\mathbb{N}$; you can count its elements off as $1, 2, 3, 4...$ but you would have to continue this process forever to count the whole set. Thus we will call $\mathbb{N}$ a **countably infinite set**, and the same term is used for any set whose cardinality equals that of $\mathbb{N}$.
Now what about the cardinality of $\mathbb{Z}$ compared to the cardinality of $\mathbb{N}$? We will not do a formal demonstration of this but let us try to understand with hands.
Well, for finite sets A and B, we counted the number of elements in each set. What mathematicians do to compare two sets A and B, is to take each element of set A and associate it with one element of set B. Hence in the case of $Card(A) = Carb(B)$, there is n exact pairing between the elements of A and the elements of B as in Figure 4.5. What happen in the case

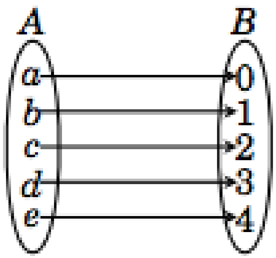


FIG. 1 – $Card(A) = Card(B)$

where $Card(A) < Card(B)$? It means that we can match all elements of A to an element of B but that it exists at least an element of B that is not paired with an element of A (see the left of Figure 2).
In the case where $Card(A) > Card(B)$, in the same way we match each element of A with an element of B but because there is less elements in B than in A, it exists at least two elements of A that are paired with the same element of B (see the right of Figure 2).

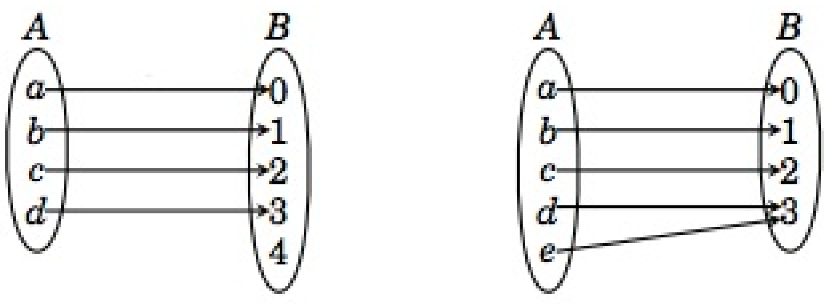


FIG. 2 – $Card(A) < Card(B)$ and $Card(A) > Card(B)$

Now, let us try to do the same between $\mathbb{N}$ and $\mathbb{Z}$ without any formalism. We will try to align element of both sets as in the following table where the first row list all the elements of $\mathbb{N}$ and the second list all the elements of $\mathbb{Z}$:
Every natural number n appears once in the first row and every integer z appears exactly once on the infinitely long second row. Thus, according to the table, given any $z \in \mathbb{Z}$, you can as-

n	0	1	2	3	4	5	6	7	...
z	0	1	-1	2	-2	3	-3	4	...

sociate it to a natural number n, thus every element of $\mathbb{Z}$ is associated to at least one element of $\mathbb{N}$. At the same time every n corresponds to one single z because of the way we constructed the table. That is to say we are in the same situation as Figure where we have an exact pairing between the elements of $\mathbb{Z}$ and the elements of $\mathbb{N}$ thus we can say that $Card(\mathbb{Z}) = Card(\mathbb{N})$.
Puzzling no? On one hand it makes sense to say that both sets have the same cardinality because both of them are infinite. However, intuitively $\mathbb{Z}$ seems twice as large as $\mathbb{N}$ because $\mathbb{Z}$ include positive and negative integers while $\mathbb{N}$ include only the positive ones.

Thus $\mathbb{N}$ and $\mathbb{Z}$ are countably infinite sets and have the same cardinality. But can we count also the elements in $\mathbb{R}$? With hands again, we will see that in fact it is not possible to count all the real numbers.
Suppose first that we *could* count all the real numbers such that we have a list of all numbers :

$$\begin{array}{c} 0.000000000.... \\ 2.123456778.... \\ 5.876543210.... \\ 7.000000000.... \\ 1.222222222.... \\ ... \end{array}$$

Now let's make a new number x. Start with 0. Now to get the first digit after the decimal point, take the first number on our list, look at its first digit after the decimal point, and change it to something different. For example, we could just add one to it, unless it is a 9 in which case we can make it a 0. Similarly, to get the second digit after the decimal point of x we take the second number on our list, look at its second digit after the decimal point, and change it as above. Continuing in this manner we can get an infinite decimal expansion for x. In the example above we would have : $x = 0.13713....$
Since we have listed all real numbers then x must be in the list. For example suppose that it is the 10th number in the list, say y. But it cannot be! Because the 10th digit of x is the 10th digit of y plus one so x cannot be y or any of the number of the list. Therefore we cannot do a list of real numbers as it is possible with $\mathbb{N}$, hence $Card(\mathbb{R}) \neq Card(\mathbb{N})$. $\mathbb{R}$ is actually called an **uncountable infinite set**.

Thus, even though $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ are all infinite sets, their cardinalities are not all the same. The sets $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality, but $\mathbb{R}$cardinality is different from that of both the other sets. Infinite sets can have different sizes!!!
We conclude on this surprising result . Next week we will see a bit more symbols, and go a bit deeper in the language of mathematics.